Plane Sextic Curves Invariant under Birational Transformations.

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Introduction.

One of the fundamental problems in the theory of birational transformations is the reduction of an algebraic curve of any given genus to its canonical form. Clebsch* has shown that the non-hyperelliptic curve of genus p can always be birationally reduced to a curve of order $p-\pi+2$, where $p=3\pi+(0, 1, 2)$; thus a general curve of genus 5 can be reduced to a sextic with five double points, and one of genus 6 to a sextic with four double points.

The groups of birational transformations belonging to normal curves of genus $p=3$ have been studied by Wiman.† He has also treated algebraic curves of genus 4, 5, 6, briefly. Miss Van Benschoten,‡ however, has studied plane curves of genus 4 in more detail, and has determined their forms and properties. The groups of birational transformations of algebraic curves of genus 5 have been studied by McKelvey,§ and those belonging to the normal curves of genus 6 have been determined by Snyder.‖

We have noted above that the canonical form of the general curve of genus either $p=5$ or $p=6$ is a sextic; moreover, the groups of birational transformations belonging to these two genera (and in particular to the canonical sextics) are especially interesting in that we are no longer concerned merely with linear transformations. It is the object of the present paper to discuss the various types of sextics which are invariant under any birational transformation, linear or non-linear.

* Clebsch, "Vorlesungen über Geometrie," Vol. I, Leipzig (1876), pp. 687, 710.

† Wiman, "Ueber die Hyperelliptischen Curven und diejenigen vom Geschlechte $p=3$," *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895).

‡ A. L. Van Benschoten, "The Birational Transformations of Algebraic Curves of Genus 4," American Journal of Mathematics, Vol. XXXI (1909).

§ J. V. McKelvey, "Groups of Birational Transformations of Algebraic Curves of Genus 5," American Journal of Mathematics, Vol. XXXIV (1912).

‖ V. Snyder, "Normal Curves of Genus 6, and their Groups of Birational Transformations," American Journal of Mathematics, Vol. XXX (1908).

The problem of finding all the algebraic plane curves invariant with respect to collineations of their plane has been solved up to and including all curves of the fifth order.* But the various types of sextics invariant under collineations of their plane have, so far, never been determined in detail. A few months ago, Vojtěch† published a memoir concerning the plane sextic curves that are invariant under periodic collineations which leave the sides of a triangle invariant. His results agree for the most part with my own, which are given in the first part of the first chapter of this paper. Dr. Vojtěch‡ omits one type invariant under a particular collineation of period 11 and includes a composite sextic among those invariant under a continuous linear group. His paper is written in Czechish and there is no abstract of it in any other language. My own results were obtained before seeing it, and, though I can not claim priority, I believe it is wisest to include this part of my results in the present paper. I am unable to comment on the methods employed by Dr. Vojtěch; our results agree with the exception of the two cases mentioned above.

After my paper was sent to the publisher, a second paper by Dr. Vojtěch appeared,§ which discusses the groups of classes (b) and (c). In a foot-note the error in the former paper is corrected. As our results agree, I believe the enumeration is complete.

In the present paper we shall discuss in the first chapter the various groups (cyclic and non-cyclic) of linear transformations which leave invariant plane sextic curves of their plane; in the second chapter we shall be concerned primarily with non-linear birational transformations, both Cremona and Riemann, which leave plane sextic curves invariant.

CHAPTER I.

LINEAR TRANSFORMATIONS.

Groups of linear transformations under which plane sextic curves remain invariant may be divided into the three following classes:

(a) Those which have as invariant points the vertices of the fundamental triangle.

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* V. Snyder, "Plane Quintic Curves which Possess a Group of Linear Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908); E. Ciani, "Le Quintiche Piane Autoproiettive," *Rend. Circ. Mat. Palermo*, Vol. XXXVI (1913).

† J. Vojtěch, "Rovinné Sextiky Invariantní při Periodických Kollineacích," Prague, 1913.

‡ Vojtěch, *loc. cit.*, p. 20, (10); p. 22, (23).

§ J. Vojtěch, "Konečné grupy kollineací a rovinné sextiky k sobě příslušne," *České akademie césaře Františka Josefa* 42, XXII (1913), 29 pp.

(b) Those which are obtained by combining transformations of class (a) with one or more transformations of the permutation group of the three homogeneous point coordinates.

(c) Those which do not belong to class (a) or to class (b), such as the simple G_{168} (Klein), which leaves invariant the Hessian of Klein's quartic.

The most simple sextic invariant under any linear transformation is the binomial sextic, of which there is only one type. The equation of this sextic is $f_6 \equiv z^6 + x^5 y = 0$. The collineation belonging to it is $\begin{pmatrix} \alpha^6 x & y & \alpha^5 z \\ x & y & z \end{pmatrix}$, where α is arbitrary. This transformation need not be periodic.

We shall now consider what sextics are left invariant by homologies, represented by $\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $\alpha^r = 1$, where r is the period. If we write the general equation of a sextic as

$$f_6 \equiv a z^6 + a_1 z^5 + a_2 z^4 + a_3 z^3 + a_4 z^2 + a_5 z + a_6 = 0,$$

where a is a constant and a_i is a binary in x, y of degree i, we find the five following types invariant under homologies of period $2 \leqq r \leqq 6$:

For $r=2$: $\quad f_6 \equiv a z^6 + a_2 z^4 + a_4 z^2 + a_6 = 0.$ $\quad (1_1)$

For $r=3$: $\quad f_6 \equiv a z^6 + a_3 z^3 + a_6 = 0.$ $\quad (2_1)$

For $r=4$: $\quad f_6 \equiv a_2 z^4 + a_6 = 0.$ $\quad$ (Derived from (1_1).) $\quad (3_1)$

For $r=5$: $\quad f_6 \equiv a_1 z^5 + a_6 = 0.$ $\quad (4_1)$

For $r=6$: $\quad f_6 \equiv a z^6 + a_6 = 0.$ $\quad$ (Derived from (2_1).) $\quad (5_1)$

(A) *General Discussion of Collineations of Type* (a).

In our further discussion of collineations of this class we shall be concerned with those which are not homologies. For collineations whose period is a power of 2, we find nine types of sextics for $r=4$, and seven for $r=8$.

For $r=4$, $C \equiv \begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4(ax^2+by^2) + z^2(cx^3y+dxy^3) + ex^6 + fx^4y^2 + gx^2y^4 + hy^6 = 0. \qquad (1_1). \qquad (6_1)$$

$$f_6 \equiv az^6 + bz^4xy + z^2(cx^4+dx^2y^2+ey^4) + fx^5y + gx^3y^3 + hxy^5 = 0. \qquad (1_1). \qquad (7_1)$$

For $r=8$, $C \equiv \begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + dz^2x^2y^2 + x^5y + xy^5 = 0. \qquad ((7_1) \text{ if } b=c=e=g=0.) \qquad (8_1)$$

This sextic is invariant under $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that it belongs to the group ($n=16$) generated by A and C. (Cf. (5_2).)

$$f_6 \equiv z^4x^2+dz^2xy^3+x^4y^2+y^6=0. \qquad ((6_1) \text{ if } b=c=e=g=0.) \qquad (9_1)$$

$$f_6 \equiv bz^4xy+z^2(x^4+y^4)+x^3y^3=0. \qquad ((7_1) \text{ if } a=d=f=h=0.) \qquad (10_1)$$

This sextic is invariant under $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that it belongs to the group ($n=16$) generated by C and A. (Cf. (6_2).)

For $r=8$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4xy+x^6+ax^4y^2+bx^2y^4+y^6=0. \qquad (3_1). \qquad (11_1)$$

$$f_6 \equiv z^4(x^2+y^2)+ax^5y+bx^3y^3+cxy^5=0. \qquad (3_1). \qquad (12_1)$$

For $r=8$, $C \equiv \begin{pmatrix} \alpha x & \alpha^7 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4x^2+cz^2x^3y+x^4y^2+y^6=0. \qquad ((6_1) \text{ if } b=d=e=g=0.) \qquad (13_1)$$

$$f_6 \equiv z^2(x^4+y^4)+fx^5y+xy^5=0. \qquad ((7_1) \text{ if } a=b=d=g=0.) \qquad (14_1)$$

There are seven types of sextics which remain invariant under collineations whose period is a power of 3: two types for $r=3$, and five for $r=9$.

For $r=3$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6+bz^4xy+z^3(cx^3+dy^3)+ez^2x^2y^2+z(fx^4y+gxy^4)+hx^6+jx^3y^3+ky^6=0. \qquad (15_1)$$

$$f_6 \equiv az^5y+bz^4x^2+cz^3xy^2+z^2(dx^3y+ey^4)+z(fx^5+gx^2y^3)+hx^4y^2+jxy^5=0. \qquad (16_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6+zxy^4+x^3y^3=0. \qquad ((15_1) \text{ if } b=c=d=e=f=h=k=0.) \qquad (17_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^3x^2y+x^6+ax^3y^3+y^6=0. \qquad (2_1). \qquad (18_1)$$

$$f_6 \equiv z^6+az^3xy^2+x^5y+bx^2y^4=0. \qquad (2_1). \qquad (19_1)$$

$$f_6 \equiv z^3(x^3+y^3)+ax^4y^2+bxy^5=0. \qquad (2_1). \qquad (20_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y+zx^5+x^4y^2=0. \qquad ((16_1) \text{ if } b=c=d=e=g=j=0.) \qquad (21_1)$$

Twelve types of sextics remain invariant under collineations whose period can be expressed as $r=2^m \cdot 3^n$, where $m \geq 1$, $n \geq 1$. For $r=6$, there are three types; for $r=12$, there are five; for $r=18$, there are two; and for $r=24$, there are two.

For $r=6$, $C\equiv\begin{pmatrix}\alpha x & \alpha^2 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^6+dz^3y^3+ez^2x^2y^2+fzx^4y+hx^6+ky^6=0. \quad ((15_1) \text{ if } b=c=g=j=0.) \quad (22_1)$$

$$f_6\equiv az^5y+bz^4x^2+ez^2y^4+gzx^2y^3+hx^4y^2=0. \quad ((16_1) \text{ if } c=d=f=j=0.) \quad (23_1)$$

For $r=6$, $C\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^6+z^3(bx^2y+cy^3)+dx^6+ex^4y^2+fx^2y^4+gy^6=0. \quad (2_1). \quad (24_1)$$

For $r=12$, $C\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+cz^3y^3+x^4y^2+y^6=0. \quad ((24_1) \text{ if } b=d=f=0.) \quad (25_1)$$

For $r=12$, $C\equiv\begin{pmatrix}\alpha x & \alpha^7 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+x^5y+gx^3y^3+xy^5=0. \quad ((5_1) \text{ or } (7_1) \text{ if } b=c=d=e=0.) \quad (26_1)$$

This sextic is invariant under $A\equiv\begin{pmatrix}y & x & z\\ x & y & z\end{pmatrix}$ also, so that its group ($n=24$) is generated by A and C. (Cf. (17_2).)

For $r=12$, $C\equiv\begin{pmatrix}\alpha x & \alpha^8 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+z^3y^3+fzx^4y+y^6=0. \quad ((22_1) \text{ if } e=h=0.) \quad (27_1)$$

This sextic is invariant under $B\equiv\begin{pmatrix}x & z & y\\ x & y & z\end{pmatrix}$ also, so that it belongs to the group ($n=24$) generated by C and B. (Cf. (18_2).)

For $r=12$, $C\equiv\begin{pmatrix}\alpha x & \alpha^{10} y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^3y^3+x^6+x^2y^4=0. \quad ((24_1) \text{ if } a=b=e=g=0.) \quad (28_1)$$

$$f_6\equiv z^6+bz^3x^2y+ex^4y^2+y^6=0. \quad ((24_1) \text{ if } c=d=f=0.) \quad (29_1)$$

For $r=18$, $C\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^3x^2y+x^6+y^6=0. \quad ((18_1) \text{ if } a=0;\ (24_1) \text{ if } a=c=e=f=0.) \quad (30_1)$$

For $r=18$, $C\equiv\begin{pmatrix}\alpha x & \alpha^{13} y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+x^5y+x^2y^4=0. \quad ((5_1);\ (19_1) \text{ if } a=0.) \quad (31_1)$$

For $r=24$, $C\equiv\begin{pmatrix}\alpha x & \alpha^{19} y & z\\ x & y & z\end{pmatrix}$,

$$f_6=z^6+x^5y+xy^5=0. \quad ((7_1) \text{ if } b=c=d=e=g=0.) \quad (32_1)$$

This equation is readily seen to be invariant under $\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that the group belonging to 32_1 is the non-cyclic G_{48}. (Cf. (24_2).)

For $r=24$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{20} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + zx^4y + y^6 = 0. \qquad ((22_1) \text{ if } d=e=h=0.) \qquad (33_1)$$

This equation is evidently invariant under $\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$ also, so that the group belonging to (33_1) is a non-cyclic G_{48}. (Cf. (25_2).)

If we consider, in the next place, collineations whose period is 5 or a power of 5, we discover three types invariant for $r=5$, and one for $r=25$.

For $r=5$, $C \equiv \begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^5x + z^4y^2 + bz^2x^3y + czx^2y^3 + dx^6 + exy^5 = 0. \qquad (34_1)$$

$$f_6 \equiv az^5y + z^3x^3 + bz^2x^2y^2 + czxy^4 + dx^5y + ey^6 = 0. \qquad (35_1)$$

$$f_6 \equiv z^4x^2 + az^3xy^2 + z^2y^4 + bzx^4y + x^3y^3 = 0. \qquad (36_1)$$

For $r=25$, $C \equiv \begin{pmatrix} \alpha x & \alpha^6 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y + x^6 + xy^5 = 0. \qquad (4_1). \qquad (37_1)$$

Collineations whose period can be expressed as $r = 5 \cdot 2^m \cdot 3^n$ leave invariant ten types. Of these, five belong to $r=10$, two to $r=15$, two to $r=20$, and one to $r=30$.

For $r=10$, $C \equiv \begin{pmatrix} \alpha^2 x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5x + z^4y^2 + x^6 = 0. \qquad ((34_1) \text{ if } b=c=e=0.) \qquad (38_1)$$

For $r=10$, $C \equiv \begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4x^2 + z^2y^4 + x^3y^3 = 0. \qquad ((36_1) \text{ if } a=b=0.) \qquad (39_1)$$

$$f_6 \equiv z^4y^2 + bz^2x^3y + x^6 + xy^5 = 0. \qquad ((34_1) \text{ if } a=c=0.) \qquad (40_1)$$

For $r=10$, $C \equiv \begin{pmatrix} \alpha x & \alpha^6 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y + x^6 + ax^4y^2 + bx^2y^4 + y^6 = 0. \qquad (4_1). \qquad (41_1)$$

For $r=10$, $C \equiv \begin{pmatrix} \alpha x & \alpha^8 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4y^2 + zx^2y^3 + x^6 = 0. \qquad ((34_1) \text{ if } a=b=e=0.) \qquad (42_1)$$

For $r=15$, $C\equiv\begin{pmatrix}\alpha x & \alpha^3 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^5y+z^3x^3+y^6=0. \qquad ((35_1) \text{ if } b=c=d=0.) \tag{43$_1$}$$

For $r=15$, $C\equiv\begin{pmatrix}\alpha x & \alpha^6 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^5y+x^6+ax^3y^3+by^6=0. \qquad (4_1). \tag{44$_1$}$$

For $r=20$, $C\equiv\begin{pmatrix}\alpha x & \alpha^6 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^5y+x^6+x^2y^4=0. \qquad (41_1) \text{ if } a=c=0.) \tag{45$_1$}$$

For $r=20$, $C\equiv\begin{pmatrix}\alpha x & \alpha^{13} y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4y^2+x^6+xy^5=0. \qquad ((34_1) \text{ if } a=b=c=0.) \tag{46$_1$}$$

For $r=30$, $C\equiv\begin{pmatrix}\alpha x & \alpha^6 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^5y+x^6+y^6=0. \qquad ((41_1) \text{ if } a=b=0.) \tag{47$_1$}$$

If the period of the collineation is 7 or a multiple of 7, we have ten types which remain invariant. To $r=7$ belong six types; to $r=14$, three; to $r=21$, one.

For $r=7$, $C\equiv\begin{pmatrix}\alpha x & \alpha^2 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4y^2+az^3x^2y+z^2x^4+xy^5=0. \tag{48$_1$}$$

$$f_6\equiv z^5y+z^4x^2+azxy^4+x^3y^3=0. \tag{49$_1$}$$

For $r=7$, $C\equiv\begin{pmatrix}\alpha x & \alpha^3 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+az^3xy^2+zx^4y+x^2y^4=0. \tag{50$_1$}$$

$$f_6\equiv z^4x^2+z^3y^3+azx^3y^2+xy^5=0. \tag{51$_1$}$$

$$f_6\equiv z^5x+az^2x^2y^2+zy^5+x^5y=0. \tag{52$_1$}$$

The last sextic is evidently invariant under $\begin{pmatrix}y & z & x\\ x & y & z\end{pmatrix}$ also. Consequently the group belonging to (52_1) is a non-cyclic G_{21}. (Cf. (31_2).)

For $r=7$, $C\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+az^2x^3y+bzx^2y^3+xy^5=0. \tag{53$_1$}$$

For $r=14$, $C\equiv\begin{pmatrix}\alpha x & \alpha^9 y & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4y^2+z^2x^4+xy^5=0. \qquad ((48_1) \text{ if } a=0.) \tag{54$_1$}$$

For $r=14$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{10} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + zx^4y + x^2y^4 = 0. \qquad ((50_1) \text{ if } a=0.) \tag{55_1}$$

For $r=14$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{11} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + z^2x^3y + xy^5 = 0. \qquad ((53_1) \text{ if } b=0.) \tag{56_1}$$

For $r=21$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{17} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5x + zy^5 + x^5y = 0. \qquad ((52_1) \text{ if } a=0.) \tag{57_1}$$

The last sextic is obviously invariant under $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$ also. Consequently the group belonging to (57_1) is a non-cyclic G_{63}. (Cf. (32_2).)

We have still to consider those collineations whose period is 11, 13, 17 or 19. For $r=11$, there are two invariant types; for $r=13$, there are two types; for $r=17$, there is one type; and for $r=19$, there is one type.

For $r=11$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{3} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5x + z^2y^4 + x^3y^3 = 0. \tag{58_1}$$

$$f_6 \equiv z^5y + z^3x^3 + x^2y^4 = 0. \tag{59_1}$$

For $r=13$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{3} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y + z^3x^3 + xy^5 = 0. \tag{60_1}$$

$$f_6 \equiv z^3x^2y + zx^5 + y^6 = 0. \tag{61_1}$$

For $r=17$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{4} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y + z^2x^4 + xy^5 = 0. \tag{62_1}$$

For $r=19$, $C \equiv \begin{pmatrix} \alpha x & \alpha^{4} y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4xy + zx^5 + y^6 = 0. \tag{63_1}$$

So far we have obtained sixty-three types which remain invariant under cyclic groups belonging to class (a); the highest cycle we obtained was of order 30. There remain a few sextics which are invariant under non-cyclic groups of this class, groups generated by two or more transformations of class (a). For these types, which are given below, n stands for the order of the group, and r for the period of the transformation.

We have for $n=4$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4(bx^2+cy^2)+z^2(dx^4+ex^2y^2+fy^4)$$
$$+gx^6+hx^4y^2+jx^2y^4+ky^6=0. \qquad (\text{Cf. } (1_2).) \qquad (64_1)$$

For $n=8$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+ex^6+fx^4y^2+gx^2y^4+hy^6=0. \quad ((6_1) \text{ if } c=d=0.) \qquad (65_1)$$

For $n=9$, $A\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$.

$$f_6\equiv az^6+z^3(cx^3+dy^3)+hx^6+jx^3y^3+ky^6=0. \quad ((15_1) \text{ if } b=e=f=g=0.) \qquad (66_1)$$

For $n=12$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6\equiv z^4x^2+z^2y^4+x^4y^2=0. \qquad ((23_1) \text{ is } a=g=0.) \qquad (67_1)$$

This sextic is evidently invariant under $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$ also, so that the group belonging to (67_1) is a G_{36}. (Cf. (21_2).)

For $n=12$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & x \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6\equiv z^6+x^6+ex^4y^2+fx^2y^4+y^6=0. \qquad ((24_1) \text{ if } b=c=0.) \qquad (68_1)$$

For $n=16$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=8$,

$$f_6\equiv z^4x^2+x^4y^2+y^6=0. \qquad ((9_1) \text{ if } d=0;\ (13_1) \text{ if } c=0.) \qquad (69_1)$$

For $n=18$, $A\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+x^6+ax^3y^3+y^6=0. \qquad (5_1). \qquad (70_1)$$

This sextic is evidently invariant under $\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that the group belonging to (70_1) is a G_{36}. (Cf. (22_2).)

For $n=24$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$, $r=12$,

$$f_6\equiv z^6+x^4y^2+y^6=0. \qquad ((25_1) \text{ if } c=0.) \qquad (71_1)$$

(B) *Collineations of Class* (b).

We shall now discuss sextics which are invariant under groups belonging to class (b). Considering first groups whose orders are powers of 2, we find that the simplest one is the non-cyclic G_4 generated by $\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$ and $\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$. The

equation belonging to this group is

$$f_6 \equiv az^6 + z^4(bx^2+cxy+by^2) + z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +kx^6+lx^5y+mx^4y^2+nx^3y^3+mx^2y^4+lxy^5+ky^6=0. \qquad (1_2)$$

This equation is equivalent to (64_1), which remains invariant under the non-cyclic G_4, generated by $\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$ and $\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$.

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $B \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6 + bz^4(x^2+y^2) + z^2(ex^4+gx^2y^2+ey^4) + kx^6+mx^4y^2+mx^2y^4+ky^6=0. \qquad (2_2)$$

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & iz \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^4(x^2+bxy+y^2)+cx^6+dx^5y+ex^4y^2+fx^3y^3+ex^2y^4+dxy^5+cy^6=0. \quad (3_1). \quad (3_2)$$

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$, $r=4$,

$$f_6 \equiv az^6+bz^4xy+cz^2(x^4+dx^2y^2+y^4)+fxy(x^4+gx^2y^2+y^4)=0. \quad (7_1). \quad (4_2)$$

For $n=16$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=8$,

$$f_6 \equiv z^6+dz^2x^2y^2+x^5y+xy^5=0. \quad (8_1). \quad (5_2)$$

$$f_6 \equiv bz^4xy+z^2(x^4+y^4)+x^3y^3=0. \quad (10_1). \quad (6_2)$$

For $n=16$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$, $r=8$,

$$f_6 \equiv z^4xy+x^6+ax^4y^2+ax^2y^4+y^6=0. \quad (11_1). \quad (7_2)$$

$$f_6 \equiv z^4(x^2+y^2)+ax^5y+bx^3y^3+axy^5=0. \quad (12_1). \quad (8_2)$$

If we consider groups whose orders are powers of 3, we have

For $n=9$, $A \equiv \begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^5y+bz^4x^2+cz^3xy^2+z^2(cx^3y+by^4)+z(ax^5+cx^2y^3) \\ +bx^4y^2+axy^5=0. \quad (16_1). \quad (9_2)$$

For groups whose orders can be expressed as $n=2^m \cdot 3^h$, where $m \geqq 1$, $h \geqq 1$, we have:

For $n=6$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^6+(bx^3+cx^2y+cxy^2+bx^3)z^3+dx^6+ex^5y+fx^4y^2 \\ +gx^3y^3+fx^2y^4+exy^5+dy^6=0. \quad (2_1). \quad (10_2)$$

For $n=6$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^6 + bz^4xy + cz^3(x^3+y^3) + ez^2x^2y^2 + fz(x^4y+xy^4) + hx^6 + jx^3y^3 + hy^6 = 0. \qquad (15_1). \qquad (11_2)$$

For $n=12$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6 \equiv az^6 + dx^6 + ex^5y + fx^4y^2 + gx^3y^3 + fx^2y^4 + exy^5 + dy^6 = 0. \qquad (5_1). \qquad (12_2)$$

For $n=12$, $A \equiv \begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6 \equiv az^6 + dz^3y^3 + ez^2x^2y^2 + fzx^4y + hx^6 + ay^6 = 0. \qquad (22_1). \qquad (13_2)$$

For $n=12$, $A \equiv \begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6 + z^4(bx^2+cy^2) + z^2(cz^4+ex^2y^2+by^4) + ax^6 + bx^4y^2 + cx^2y^4 + ay^6 = 0. \qquad (64_1). \qquad (14_2)$$

For $n=18$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $B \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv z^6 + bz^3(x^3+y^3) + x^6 + gx^3y^3 + y^6 = 0. \qquad (10_2) \text{ or } (66_1). \qquad (15_2)$$

For $n=18$, $A \equiv \begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$, $B \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + bz^4xy + cz^3(x^3+y^3) + ez^2x^2y^2 + bz(x^4y+xy^4) + x^6 + cx^3y^3 + y^6 = 0. \qquad (11_2). \qquad (16_2)$$

For $n=24$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^7 y & z \\ x & y & z \end{pmatrix}$, $r=12$,

$$f_6 \equiv z^6 + x^5y + x^3y^3 + xy^5 = 0. \qquad (26_1). \qquad (17_2)$$

For $n=24$, $A \equiv \begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^8 y & z \\ x & y & z \end{pmatrix}$, $r=12$,

$$f_6 \equiv z^6 + z^3y^3 + zx^4y + y^6 = 0. \qquad (27_1). \qquad (18_2)$$

For $n=24$, $A \equiv \begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$, $B \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + z^2x^2y^2 + x^6 + y^6 = 0. \qquad (13_2). \qquad (19_2)$$

(19_2) permits necessarily, in addition, $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, giving us for its group $n=72$, instead of $n=24$.

For $n=24$, $A \equiv \begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $B \equiv \begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $D \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + bz^4(x^2+y^2) + bz^2(x^4+ex^2y^2+y^4) + x^6 + bx^4y^2 + bx^2y^4 + y^6 = 0. \qquad (14_2). \qquad (20_2)$$

For $n=36$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4x^2+z^2y^4+x^4y^2=0. \qquad (67_1). \qquad (21_2)$$

For $n=36$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+x^6+ax^3y^3+y^6=0. \qquad (70_1). \qquad (22_2)$$

For $n=24$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+x^6+ex^4y^2+ex^2y^4+y^6=0. \qquad (68_1). \qquad (23_2)$$

For $n=48$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{19} y & z \\ x & y & z \end{pmatrix}$, $r=24$,

$$f_6\equiv z^6+x^5y+xy^5=0. \qquad (32_1). \qquad (24_2)$$

For $n=48$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{20} y & z \\ x & y & z \end{pmatrix}$, $r=24$,

$$f_6\equiv z^6+zx^4y+y^6=0. \qquad (33_1). \qquad (25_2)$$

For $n=54$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$,

$D\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+bz^3(x^3+y^3)+x^6+bx^3y^3+y^6=0. \qquad (15_2). \qquad (26_2)$$

For $n=72$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$, $D\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+az^2x^2y^2+x^6+y^6=0. \qquad (22_2). \qquad (27_2)$$

For $n=216$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$D\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6\equiv x^6+y^6+z^6=0. \qquad (27_2). \qquad (28_2)$$

For groups whose orders are multiples of 5, we have:

For $n=10$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=5$,

$$f_6\equiv az^5(x+y)+bx^6+cx^5y+dx^4y^2+ex^3y^3+dx^2y^4+cxy^5+by^6=0. \qquad (4_1). \qquad (29_2)$$

For $n=10$, $A\equiv\begin{pmatrix} z & y & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=5$,

$$f_6\equiv az^5y+z^3x^3+bz^2x^2y^2+czxy^4+ax^5y+by^6=0. \qquad (35_1). \qquad (30_2)$$

Finally, for groups whose orders are multiples of 7, we have:

For $n=21$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=7$,

$$f_6\equiv z^5x+az^2x^2y^2+zy^5+x^5y=0. \qquad (52_1). \qquad (31_2)$$

For $n=63$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{17} y & z \\ x & y & z \end{pmatrix}$, $r=21$,

$$f_6\equiv z^5x+zy^5+x^5y=0. \qquad (57_1). \qquad (32_2)$$

(c) *Collineations of Class* (c).

The well-known groups which belong to class (c), namely the G_{168} and the G_{360}, are both simple groups. The G_{168} was derived by Klein* through consideration of transformations of the seventh order of elliptic functions. The sextic which is invariant under the G_{168} is the Hessian of the Klein quartic,

$$C_4\equiv x^3y+y^3z+z^3x=0,$$

viz.,

$$f_6\equiv z^5y-5z^2x^2y^2+zx^5+xy^5=0. \qquad (1_3)$$

It is thus seen to be a particular case of (31_2).

The simple group G_{360} was first exhibited by Valentiner,† and it has been studied in some detail by him and also by A. Wiman.‡ The simplest curve belonging to the G_{360} is

$$f_6\equiv 9z^5y+10z^3x^3-45z^2x^2y^2-135zxy^4+9x^5y+27y^6=0. \qquad (2_3)$$

This is seen to be a particular case of (35_1).

The G_{360} is made up of forty-five collineations of period 2, eighty of period 3, ninety of period 4, and one hundred and forty-four of period 5.

The remaining groups of class (c) are the regular body groups not belonging to class (a) or class (b).

From the tetrahedral G_{12}, which leaves the two binary forms $xy(x^4-y^4)$, $x^4+2i\sqrt{3}x^2y^2+y^4$ invariant, we obtain the sextic

$$f_6\equiv az^6+bz^2(x^4+2i\sqrt{3}x^2y^2+y^4)+cxy(x^4-y^4)=0, \qquad (3_3)$$

invariant under G_{24}, defined by

$$\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}, \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}, \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}, \text{ and } \begin{pmatrix} x+iy & x-iy & \alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha=\sqrt{i+\sqrt{3}}.$$

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* F. Klein, "Ueber die Transformation siebenter Ordnung der elliptischen Funktionen," *Mathematische Annalen*, Vol. XIV, pp. 428–471.

† Valentiner, "De endelige Transformations Gruppers Theori."

‡ A. Wiman, "Ueber eine einfache Gruppe von 360 ebenen Collineationen," *Mathematische Annalen*, Vol. XLVII, pp. 531–556.

If $b=0$, we have the sextic

$$f_6 \equiv az^6 + cxy(x^4 - y^4) = 0 \tag{4_3}$$

invariant under G_{72}, defined by the above transformations and also

$$\begin{pmatrix} x & y & \beta z \\ x & y & z \end{pmatrix}, \qquad \beta^3 = 1.$$

The form (24_2) has also the transformation

$$\begin{pmatrix} x+\alpha^3 y & \alpha^3(x-\alpha^3 y) & \sqrt{2}\alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha^{24} = 1,$$

making a G_{144}. Consequently we have invariant under a G_{144}

$$f_6 \equiv xy(x^4 + y^4) + z^6 = 0 \tag{5_3}$$

Similarly (26_2), if $b=-10$, has also the transformation

$$\begin{pmatrix} x+y+z & x+\alpha y+\alpha^2 z & x+\alpha^2 y+\alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha^3 = 1,$$

so that

$$f_6 \equiv x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3) = 0 \tag{6_3}$$

is invariant under a G_{216}.

The same substitution may be applied to

$$f_6 \equiv x^6 + y^6 + z^6 + a(x^3y^3 + y^3z^3 + z^3x^3) - \frac{a+10}{2}[xyz(x^3+y^3+z^3) + 3x^2y^2z^2] = 0, \qquad (16_2), \tag{7_3}$$

a curve invariant, therefore, under a G_{36}.

The icosahedron group leaves invariant the sextics already found by Klein,*

$$f_6 \equiv (x^5 + y^5)z + (a-1)x^3y^3 + (2+3a)x^2y^2z^2 + (3a-8)xyz^4 + az^6 = 0. \tag{8_3}$$

The G_{60} is defined by

$$\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}, \begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}, \quad \alpha^5 = 1,$$

and

$$\begin{pmatrix} (\alpha^2+\alpha^3)x + (\alpha+\alpha^4)y + 2z & (\alpha+\alpha^4)x + (\alpha^2+\alpha^3)y + 2z & x+y+z \\ x & y & z \end{pmatrix}.$$

CHAPTER II.

NON-LINEAR TRANSFORMATIONS.

In any attempt to determine the types of sextics which are invariant under non-linear birational transformations, we need not consider any curve whose genus is higher than 7. For any such sextic remains invariant only under

* Klein, "Ikosaeder," p. 215, Eq. (9), and p. 218, Eq. (14).

linear transformations.* On the other hand, any curve of genus zero may be reduced by birational transformations to a straight line; a curve of genus 1 to a non-singular cubic; a curve of genus 2 to a quartic with one double point; a non-hyperelliptic curve of genus 3 to a non-singular quartic; and a non-hyperelliptic curve of genus 4 to a quintic with two double points. The groups belonging to the canonical form of each of these genera have been determined. In this paper only one particular case of sextics of genus less than 5 will be discussed, namely that of the sextic of genus 3, whose seven double points are distinct and independent. We shall consider first, however, sextics whose genus is 7, 6, or 5.

The only non-linear Cremona transformations under which sextics of genus 7 remain invariant, are quadratic transformations; and of these sextics there are various types depending upon the configuration of the double points. The following cases will be considered in order:

(a) An f_6 with three non-collinear P_2's.
  1. All three P_2's distinct.
  2. Tacnode (or node cusp) and a P_2 not on the tacnodal tangent.
  3. Oscnode (or tacnode cusp).
  4. Triple point.

(b) An f_6 with three collinear P_2's.
  1. All three distinct.
  2. Tacnode and a P_2 on the tacnodal tangent.
  3. All three coincident.

(a) 1. If we take double points as the vertices of the triangle of reference, the most general sextic having three distinct non-collinear double points is

$$f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3)+z^2(hx^4+jx^3y+kx^2y^2 +lxy^3+my^4)+z(nx^4y+px^3y^2+qx^2y^3+rxy^4)+sx^4y^2+tx^3y^3+ux^2y^4=0.$$

If f_6 is to be invariant under a quadratic transformation of the first kind,† certain relations must be true among the coefficients of f_6, so that we have:

For $r=2$, $T \equiv \begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) +cx^4y^2+bx^3y^3+ax^2y^4=0. \tag{1}$$

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* C. Küpper, "Ueber das Vorkommen von linearen Schaaren $g^2{}_n$ auf Kurven n^{ter} Ordnung," *Sitzungsberichte der Böhmischen Gesellschaft* (Prag, 1892), pp. 264–272; V. Snyder, "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), p. 10.

† C. A. Scott, "Modern Analytical Geometry" (1894), § 233.

If $a=c$, $d=g$, $e=f$, we have for $n=4$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6\equiv & z^4(ax^2+bxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ & +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(dx^4y+ex^3y^2+ex^2y^3+dxy^4) \\ & +ax^4y^2+bx^3y^3+ax^2y^4=0. \end{aligned} \tag{2}$$

If, in (2), $b=d$, $a=h$, $e=j$, we have for $n=12$,

$$T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix},\quad A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix},\quad B\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix},$$

$$\begin{aligned} f_6\equiv & z^4(ax^2+bxy+ay^2)+z^3(bx^3+ex^2y+exy^2+by^3) \\ & +z^2(ax^4+ex^3y+kx^2y^2+exy^3+ay^4)+z(bx^4y+ex^3y^2+ex^2y^3+bxy^4) \\ & +ax^4y^2+bx^3y^3+ax^2y^4=0. \end{aligned} \tag{3}$$

If in (3), $b=e=0$, we have for $n=72$,

$$T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix},\quad A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix},\quad B\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix},\quad C\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix},\quad D\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix},$$

$$f_6\equiv z^4(x^2+y^2)+z^2(x^4+kx^2y^2+y^4)+x^4y^2+x^2y^4=0. \tag{4}$$

(a) 2. The most general equation of a sextic of this type is

$$\begin{aligned} f_6\equiv & az^4y^2+z^3y(bx^2+cxy+dy^2)+z^2(ex^4+fx^3y+gx^2y^2+hxy^3+jy^4) \\ & +zx(kx^4+lx^3y+mx^2y^2+nxy^3+py^4)+x^2(qx^4+rx^3y+sx^2y^2+txy^3+vy^4)=0, \end{aligned}$$

which has a tacnode at (0, 0, 1) with $y=0$ as tacnodal tangent, and also a double point at (0, 1, 0).

If f_6 is to be invariant under a quadratic transformation, the transformation must be of the second kind* and certain relations must hold between the coefficients of the sextic. When these conditions are satisfied, we have for $r=2$,

$$T\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix},$$

$$\begin{aligned} f_6\equiv & az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ & +zx(kx^4+lx^3y+mx^2y^2+lxy^3+ky^4) \\ & +x^2(qx^4+rx^3y+sx^2y^2+rxy^3+qy^4)=0. \end{aligned} \tag{5}$$

If in (5), $c=f=k=m=r=0$, we have:

For $n=4$, $T\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6\equiv & az^4y^2+byz^3(x^2+y^2)+z^2(ex^4+gx^2y^2+ey^4) \\ & +lzx^2y(x^2+y^2)+x^2(qx^4+sx^2y^2+qy^4)=0. \end{aligned} \tag{6}$$

(a) 3. The most general equation of a sextic which has an oscnode at (0, 0, 1) is

* C. A. Scott, "Modern Analytical Geometry" (1894), § 234, (2).

$$f_6 \equiv az^2(mx^2-yz)^2+(mx^2-yz)(bxyz^2+cx^3z) \\ +dy^3z^3+z^2y^2\phi_2(x,y)+zy\phi_4(x,y)+\phi_6(x,y)=0.$$

The quadratic inversion which will leave this sextic invariant must be of the third kind.* Consequently we have:

For $r=2$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^2(mx^2-yz)^2+cx^3z(mx^2-yz)+yz(mx^2-yz)\phi_2(x,y)+\phi_6(x,y)=0. \quad (7)$$

If in (7) $c=0$, and $\phi_2(x,y)=\phi_1(x^2,y^2)$, $\phi_6(x,y)=\phi_3(x^2,y^2)$:

For $n=4$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^2(mx^2-yz)^2+yz(mx^2-yz)\phi_1(x^2,y^2)+\phi_3(x^2,y^2)=0. \quad (8)$$

(a) 4. If we take the triple point as one vertex $(0,0,1)$ of the triangle of reference, we have, as the most general equation of a sextic with one triple point,

$$f_6\equiv z^3\phi_3(x,y)+z^2\phi_4(x,y)+z\phi_5(x,y)+\phi_6(x,y)=0.$$

It is readily seen that this sextic is invariant under no non-linear Cremona transformation. The linear ones under which it remains invariant will be seen to belong to classes (a) and (b) as listed in Chapter I.

For $r=2$, $C\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^3(ax^3+bx^2y+bxy^2+ay^3)+z^2(cx^4+dx^3y+ex^2y^2+dxy^3+cy^4) \\ +z(fx^5+gx^4y+hx^3y^2+hx^2y^3+gxy^4+fy^5)+jx^6+kx^5y \\ +mx^4y^2+nx^3y^3+mx^2y^4+kxy^5+jy^6=0. \quad \text{(Equivalent to } (1_1).)$$

For $r=3$, $C\equiv\begin{pmatrix} \alpha x & y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^3(ax^3+dy^3)+yz^2(fx^3+jy^3)+y^2z(mx^3+qy^3)+rx^6+sx^3y^3+ty^6=0. \quad (2_1).$$

For $r=3$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^3(ax^3+dy^3)+ez^2x^2y^2+z(fx^4y+gxy^4)+hx^6+jx^3y^3+ky^6=0. \quad (15_1).$$

For $n=6$, $C\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z\end{pmatrix}$, $r=3$,

$$f_6\equiv z^3(ax^3+bx^2y+bxy^2+ay^3)+jx^6 \\ +kx^5y+mx^4y^2+nx^3y^3+mx^2y^4+kxy^5+jy^6=0. \quad (10_2).$$

For $n=6$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$, $r=3$,

$$f_6\equiv cz^3(x^3+y^3)+ez^2x^2y^2+fz(x^4y+xy^4)+hx^6+jx^3y^3+hy^6=0. \quad (11_2).$$

* C. A. Scott, "Modern Analytical Geometry" (1894), § 234, (3).

For $n=9$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv z^3(cx^3+dy^3)+hx^6+jx^3y^3+ky^6=0. \qquad (66_1).$$

For $n=18$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv bz^3(x^3+y^3)+x^6+gx^3y^3+y^6=0. \qquad (15_2).$$

For $n=18$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv cz^3(x^3+y^3)+ez^2x^2y^2+x^6+cx^3y^3+y^6=0. \qquad (16_2).$$

For $n=54$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$,

$B\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $D\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv bz^3(x^3+y^3)+x^6+bx^3y^3+y^6=0. \qquad (26_2).$$

As can be seen immediately, no sextic of genus 7, which possesses three collinear double points, coincident or distinct, remains invariant under non-linear Cremona transformations.

(b) 1. The most general equation of a sextic with three distinct double points (at $(0, 1, 0)$, $(m, 1, 0)$, $(1, 0, 0)$) is

$$f_6\equiv az^6+z^5\phi_1(x, y)+z^4\phi_2(x, y)+z^3\phi_3(x, y)+z^2\phi_4(x, y) \\ +xyz(x-my)f_2(x, y)+x^2y^2(x-my)^2=0.$$

If $\phi_1=\phi_3=f_2=0$, we have for $r=2$, $\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4\phi_2(x, y)+z^2\phi_4(x, y)+x^2y^2(x-my)^2=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2 \\ +fxy^3+ey^4)+x^2y^2(x+y)^2. \qquad (1_2).$$

(b) 2. If two of the double points are coincident and the other distinct, we have, as the equation of the sextic,

$$f_6\equiv z^4\phi_2(x, y)+z^3yf_2(x, y)+z^2y\phi_3(x, y)+zy^2f_3(x, y)+y^2\phi_4(x, y)=0.$$

This sextic has a tacnode at $(1, 0, 0)$ and a double point at $(0, 0, 1)$.

If $f_2=f_3=0$, the sextic is seen to belong to type (1_1), which is invariant under $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$.

For $r=4$, $A\equiv\begin{pmatrix} x & y & iz \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4\phi_2(x, y)+y^2\phi_4(x, y)=0. \qquad (3_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+z^2xy(cx^2+ey^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (6_1).$$

For $n=4$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+z^2y^2(dx^2+fy^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (64_1).$$

For $n=8$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (65_1).$$

(b) 3. The most general equation of a sextic which has three coincident collinear double points (at (1, 0, 0)) is

$$f_6\equiv az^6+z^5f_1(x, y)+z^4f_2(x, y)+z^3f_3(x, y)+z^2f_4(x, y)+zy^2\,\phi_3(x, y)+ty^6=0.$$

For $r=2$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, we have

$$f_6\equiv az^6+cz^5y+z^4(dx^2+ey^2)+z^3y(gx^2+hy^2)$$
$$+z^2(jx^4+lx^2y^2+ny^4)+zy^3(px^2+sy^2)+ty^6=0. \qquad (1_1).$$

For $r=3$, $A\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+bz^5x+dz^4x^2+z^3(fx^3+hy^3)+z^2x(jx^3+my^3)+pzx^2y^3+ty^6=0. \qquad (2_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+cz^5y+ez^4y^2+hz^3y^3+z^2(jx^4+ny^4)+szy^5+ty^6=0. \qquad (3_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(dx^2+ey^2)+z^2xy(kx^2+my^2)+ty^6=0. \qquad (6_1).$$

For $r=6$, $A\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+bz^5x+dz^4x^2+fz^3x^3+jz^2x^4+ty^6=0. \qquad (5_1).$$

For $n=4$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4(dx^2+ey^2)+z^2(jx^4+lx^2y^2+ny^4)+ty^6=0. \qquad (64_1).$$

For $n=12$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha^4x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6\equiv z^6+dz^4x^2+jz^2x^4+ty^6=0. \qquad (68_1).$$

The only non-linear Cremona transformations under which sextics of genus 6* remain invariant are quadratic transformations, and of these sextics there are various types depending upon the configuration of the double points. The following cases will be considered:

(α) No three collinear.
  1. All distinct.
  2. One tacnode and two other P_2's.
    α) 2 P_2's distinct.
    β) 2 P_2's forming a second tacnode.
  3. Oscnode and one other P_2.
  4. Triple point and one other P_2.
  5. Four coincident.

(β) Three collinear.
  1. All distinct.
  2. Two tacnodes (one on tangent of the other).
  3. Three consecutive.
  4. Four consecutive.

(α) 1. The most general equation of a sextic with four double points (at (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)) which is invariant under a quadratic transformation of the first kind, $\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, is

$$\begin{aligned} f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) \\ +cx^4y^2+bx^3y^3+ax^2y^4=0, \end{aligned} \tag{1}$$

where $2(a+b+c+d+e+f+g+h+j)+k=0$.

If it is invariant under a harmonic homology also, we have:

For $n=4$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6 \equiv z^4(ax^2+bxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+zxy(dx^3+ex^2y+exy^2+dy^3) \\ +ax^4y^2+bx^3y^3+ax^2y^4=0, \end{aligned} \tag{2}$$

where $2(2a+b+2d+2e+h+j)+k=0$.

For $n=12$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A=\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} z & y & x \\ x & y & z \end{pmatrix}$,

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* Cf. Snyder, "Normal Curves of Genus 6, and Their Groups of Birational Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908).

$$f_6 \equiv z^4(ax^2+dxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ +z^2(ax^4+ex^3y+kx^2y^2+exy^3+ay^4)+zxy(dx^3+ex^2y+exy^2+dy^3) \\ +ax^4y^2+dx^3y^3+ax^2y^4=0. \tag{3}$$

where $2(3a+3d+3e)+k=0$.

For $n=120$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$B\equiv\begin{pmatrix} z & y & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & (x-y) & (x-z) \\ x & y & z \end{pmatrix}$,

$a=2$, $d=-2$, $e=-1$, $k=6$, and we have

$$f_6 \equiv 2z^4(x^2-xy+y^2)-z^3(2x^3+x^2y+xy^2+2y^3) \\ +z^2(2x^4-x^3y+6x^2y^2-xy^3+2y^4)-zxy(2x^3+x^2y+xy^2+2y^3) \\ +2x^2y^2(x^2-xy+y^2)=0. \tag{4}$$

(α) 2, α). This case appears as a subcase of (a) 2. The sextic in (a) 2 with a tacnode at (0, 0, 1) with $y=0$ as tangent, and a double point at (0, 1, 0), invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, was

$$f_6 \equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +zx(kx^4+lx^3y+mx^2y^2+lxy^3+ky^4)+x^2(qx^4+rx^3y+sx^2y^2+rxy^3+qy^4)=0.$$

If f_6 is to have another double point, we may take it at (1, 1, 0). Then the coefficients of f_6 must satisfy the conditions $2q+2r+s=0$, $2k+2l+m=0$. Imposing these conditions, we have as the equation of the sextic invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, and which has a tacnode and two double points,

$$f_6 \equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +x(x-y)^2[(kz+qx)(x+y)^2+xy(lz+rx)]=0.$$

For $n=4$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $c=f=k=l=r=0$, and we have

$$f_6 \equiv az^4y^2+byz^3(x^2+y^2)+z^2(ex^4+gx^2y^2+ey^4)+qx^2(x-y)^2(x+y)^2=0.$$

For $n=8$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $b=0$, and we have

$$f_6 \equiv az^4y^2+z^2(ex^4+yx^2y^2+ey^4)+qx^2(x-y)^2(x+y)^2=0.$$

(α) 2, β) The most general equation of a sextic of genus 6 having two tacnodes (neither one of which is on the tacnodal tangent of the other) is

$$f_6 \equiv az^4y^2+z^3y\phi_2(x,y)+z^2\phi_4(x,y)+zx^2\phi_3(x,y)+x^4f_2(x,y)=0.$$

This sextic remains invariant under no non-linear Cremona transformations, but the linear ones are:

For $r=2$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^3y(bx^2+cxy+dy^2)+z^2(ex^4+fx^3y+gx^2y^2+cxy^3+ay^4)$$
$$+zx^2(kx^3+lx^2y+fxy^2+by^3)+px^6+kx^5y+ex^4y^2=0. \quad (1)$$

For $n=4$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^3y(bx^2+dy^2)+z^2(ex^4+gx^2y^2+ay^4)$$
$$+zx^2(lx^2y+by^3)+px^6+ex^4y^2=0. \quad (2)$$

For $n=8$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2(ex^4+gx^2y^2+ay^4)+px^6+ex^4y^2=0. \quad (3)$$

(α) 3. The most general equation of a sextic with oscnode at (0, 0, 1) and another P_2 at (0, 1, 0), invariant under a quadratic transformation $\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$ is,

$$f_6\equiv az^2(mx^2-yz)^2+cx^3z(mx^2-yz)+yz(mx^2-yz)\phi_2(x, y)+x^2\phi_4(x, y)=0. \quad (1)$$

For $n=4$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^2(mx^2-yz)^2+yz(mx^2-yz)(ax^2+by^2)+x^2\phi_2(x^2, y^2)=0. \quad (2)$$

(α) 4. The most general equation of a sextic with a triple point (at (0, 0, 1)) and a P_2 at (0, 1, 0) is

$$f_6\equiv z^3\phi_3(x, y)+z^2\phi_4(x, y)+zxf_4(x, y)+x^2yf_3(x, y)=0,$$

which has a simple point at (1, 0, 0).

If $f_3(x, y)=\phi_3(y, x)$, $f_4(x, y)=\phi_4(y, x)$, we have:

For $r=2$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3\phi_3(x, y)+z^2\phi_4(x, y)+zx\phi_4(y, x)+x^2y\phi_3(y, x)=0. \quad (1)$$

For $n=6$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$.

$$f_6\equiv z^3\phi_3(x, y)+x^2y\phi_3(y, x)=0. \quad (2)$$

For $n=8$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3x^3+z^2(ex^4+jy^4)+zx(jx^4+ey^4)+x^2y^4=0. \quad (3)$$

For $n=18$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} \alpha^4 x & \alpha y & z \\ x & y & z\end{pmatrix}$, $r=9$,

$$f_6\equiv z^3(ax^3+by^3)+x^2y(bx^3+ay^3)=0. \tag{4}$$

(α) 5. The only sextic having four coincident P_2's which is invariant under any transformation is

$$f_6\equiv az^3xy^2+z^2y^2(bx^2+cy^2)+zx(dx^4+dx^2y^2+ey^4)+y^2(fx^4+gx^2y^2+hy^4)=0.$$

This is invariant under the single harmonic homology, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$.

(β) 1. If a sextic with three collinear P_2's and one other one, is inverted as to a triangle of the double points, the third double point on one of the sides of the triangle becomes a tacnode; and the latter case, which is as general as the given one, has been discussed in (α) 2, α).

(β) 2. The most general equation of a sextic with two tacnodes, one of which is on the tacnodal tangent of the other, is

$$f_6\equiv az^4y^2+yz^3\phi_2(x, y)+z^2\phi_4(x, y)+y^2z\phi_3(x, y)+y^4f_2(x, y)=0,$$

where the tacnode (1, 0, 0) is on the tacnodal tangent, $y=0$, of the other tacnode (0, 0, 1).

This sextic is invariant only under the linear transformations of the non-cyclic G_4.

For $r=2$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+bz^3xy^2+z^2(cx^4+dx^2y^2+ey^4)+zxy^2(fx^2+gy^2)+(hx^2+jy^2)y^4=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2(cx^4+dx^2y^2+ey^4)+y^4(hx^2+jy^2)=0.$$

(β) 3. The most general equation of a sextic with three consecutive collinear double points (at (0, 0, 1)) and one other double point (at (0, 1, 0)), is

$$f_6\equiv z^4y\phi_1(x, y)+z^3y\phi_2(x, y)+z^2y\phi_3(x, y)+zxyf_3(x, y)+x^2\phi_4(x, y)=0.$$

This sextic is invariant under linear transformations of the non-cyclic G_4.

For $r=2$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+bz^3xy^2+z^2y^2(cx^2+dy^2)+zxy^2(fx^2+gy^2)+x^2(hx^4+jx^2y^2+ky^4)=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2y^2(cx^2+dy^2)+x^2(hx^4+jx^2y^2+ky^4)=0.$$

(β) 4. If all four double points are consecutive, they must lie on a conic, and the sextic is invariant only under a harmonic homology with center on the

tangent and axis through the double point. (Cf. Snyder, "Normal Curves of Genus 6," p. 331.)

The only non-linear Cremona transformations under which sextics of genus 5 remain invariant are quadratic transformations; of these sextics there are various types, depending upon the configuration of the double points. These various types appear as particular cases of sextics of genus either 7 or 6, and they will be discussed in the following order:

(1) An f_6 with five distinct P_2's.
(2) An f_6 with one tacnode and
  a) Three distinct P_2's (general).
  b) One other tacnode with a P_2 at point of intersection of the tacnodal tangents.
  c) Three distinct collinear P_2's, one of which is on the tacnodal tangent.
  d) Three coincident collinear P_2's.
(3) An f_6 with an oscnode and two P_2's.

(1) From (a) 1 we have as the equation of the sextic of genus 7 which is invariant under $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$,

$$
\begin{aligned}
f_6 \equiv\, & z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) \\
& +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) \\
& +cx^4y^2+bx^3y^3+ax^2y^4=0.
\end{aligned}
$$

If this sextic is to have two more double points which are interchanged by C, we need impose merely the conditions that it have, as a double point, $(\alpha, \beta, 1)$, not one of the invariant points; and then our sextic, since it remains invariant under the involution C, must have, as a fifth double point, $\left(\frac{1}{\alpha}, \frac{1}{\beta}, 1\right)$, the image of $(\alpha, \beta, 1)$ under C.

If the sextic (1) of (α) 1 is to have a fifth double point and yet remain invariant under $\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$, the fifth double point must be $(1, 1, -1)$. If $(1, 1, -1)$ is to be a double point of (α) 1, (1), we must have $d+e+f+g=0$. Imposing this condition, we have:

For $r=2$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$,

$$
\begin{aligned}
f_6 \equiv\, & ax^2(y^2-z^2)^2+bxy(z^2-xy)^2+cy^2(z^2-x^2)^2+dz(x^3-y^3)(z^2-xy) \\
& +eyz(x^2-y^2)(z^2-x^2)+fyz(yz^3-x^3)(x-y)+hz^2(x^2-y^2)^2 \\
& +jz^2xy(x-y)^2=0. \qquad (1)
\end{aligned}
$$

If $d=e=f=0$, (1) is invariant for $n=4$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$.

(2), a) If the sextic of (a) 2 which is invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z\end{pmatrix}$ is to have two more double points, which are interchanged by C, we need impose merely the conditions that $f_6=0$ has as one double point $(\alpha, \beta, 1)$ not on the fundamental system of the transformation, and then it will necessarily have as another double point $\left(\alpha, \frac{\alpha^2}{\beta}, 1\right)$, the image of $(\alpha, \beta, 1)$ under C, since f_6 remains invariant under C.

If the sextic of (α) 2, α), which has $(0, 0, 1)$ as a tacnode with $y=0$ as the tangent, and $(0, 1, 0)$ and $(1, 1, 0)$ as two other double points, is to have a fifth double point and yet remain invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z\end{pmatrix}$, this fifth double point must be on the invariant conic $(x+y)(x-y)=0$, and certain further restrictions must be imposed on the coefficients of $f_6=0$.

(2), b) If, on the sextic of (a) 2 which is invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z\end{pmatrix}$, we impose the condition that it shall have a double point at $(1, 0, 0)$, the double point at $(0, 1, 0)$ becomes a tacnode with tangent $z=0$, and we have:

For $r=2$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4)$$
$$+x^2yz(nx^2+mxy+ny^2)+rx^4y^2=0.$$

This is also a particular case of (α) 2, β). If $a=e=r$, $b=n$, $f=c=m$, our sextic is invariant under $A\equiv\begin{pmatrix} x & z & y \\ x & y & z\end{pmatrix}$ also, therefore under a group of order $n=8$.

If, in addition, $c=0$, $f_6=0$ is invariant under $\begin{pmatrix} -x & y & z \\ x & y & z\end{pmatrix}$ also, therefore under a group of order $n=16$.

If, furthermore, $b=0$, we have:

For $n=32$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z\end{pmatrix}$, $D\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4y^2+z^2(x^4+gx^2y^2+y^4)+x^4y^2=0.$$

(2), c) If we impose on the sextic of (b) 1, which has three collinear double points (at $(0, 1, 0)$, $(m, 1, 0)$, $(1, 0, 0)$), the conditions that it shall have a tacnode at $(0, 0, 1)$ with $x=my$ as the tacnodal tangent, we have as the equation of the sextic (when $m=1$),

$$f_6\equiv(x-y)^2(z^4+x^2y^2)+z(x-y)f_2(x, y)(z^2+xy)+z^2f_4(x, y)=0.$$

It is readily seen that this sextic is invariant under $C\equiv\begin{pmatrix} xz & yz & xy \\ x & y & z\end{pmatrix}$.

If $f_2=0$, our sextic is invariant under a group, whose order is $n=4$, generated by C and $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$.

(2), d) If we impose on the sextic of (b) 3, which has three coincident collinear double points at (1, 0, 0), the conditions that it shall have (0, 0, 1) as a tacnode with $y=0$ as the tacnodal tangent, we have:

$$f_6\equiv z^4y^2+yz^3\phi_2(x, y)+z^2\phi_4(x, y)+y^3z\phi_2(x, y)+ky^6=0,$$

which is invariant under $C\equiv\begin{pmatrix} xz & yz & y^2 \\ x & y & z\end{pmatrix}$. f_6 is seen to be a particular case of the sextic in (β) 2.

If $\phi_2=0$, we have:

For $n=4$, $C\equiv\begin{pmatrix} xz & yz & y^2 \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4y^2+z^2\phi_4(x, y)+ky^6=0.$$

(3) If the sextic of (a) 3, which is invariant under $C\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z\end{pmatrix}$, is to have two more double points, which are interchanged by C, we need impose merely the condition that $f_6=0$ shall have as one double point (α, β, 1) not on the fundamental system of the inversion; it will then necessarily have as another double point the image of (α, β, 1) under C, since f_6 remains invariant under C.

From (α) 3 we find that the sextic with oscnode at (0, 0, 1) with $y=0$ as tangent, and a double point at (0, 1, 0), invariant under $C\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z\end{pmatrix}$, is

$$f_6\equiv az^2(mx^2-yz)^2+cx^3z(mx^2-yz)+yz(mx^2-yz)\phi_2(x, y)+x^2\phi_4(x, y)=0.$$

If this sextic is to remain invariant under C and yet have an additional double point, this double point must be on the invariant conic $mx^2-2yz=0$, and certain further conditions must be imposed upon the coefficients of $f_6=0$.

We shall now consider, in conclusion, those sextics of genus 3 which have seven distinct double points.* If we let

$$f_6(x)=0 \tag{1)}$$

be the equation of a sextic with seven distinct double points and

$$\phi_i=0 \qquad (i=1, 2, 3) \tag{2)}$$

* See p. 323 above.

the equations of three linearly independent adjoint cubics, then, if we eliminate x between 1) and 2), we shall obtain

$$F_4(y)=0 \tag{3}$$

as the equation of the non-singular quartic into which our sextic is transformed by means of the adjoint cubics. If in 3) we make the substitution 2), we obtain $X(x)\equiv f_6(x)\cdot M_6(x)=0$, where $M_6=0$ is a sextic which has double points at the same seven points as $f_6=0$. If we consider $f_6=0$ as the locus of the eighth basis point ξ of a pencil of cubics of the net, then $M=0$ is traced by the ninth basis point ξ' of the pencil. Therefore $M=0$ is the image of $f_6=0$ under the Geiser transformation which has the seven double points as fundamental points. $M=0$ may be: 1), distinct from $f=0$; 2), coincident with $f=0$ (not pointwise); or 3), pointwise coincident. Case 1) does not concern us, as we are interested only in invariant sextics. However, we shall need to consider both 2) and 3).

Case 2) If $f_6=0$ is to remain invariant (not pointwise) under the Geiser transformation, then, if we choose any point on $f_6=0$, not on the invariant sextic of the transformation, as an eighth basis point ξ of a pencil of adjoint cubics, the ninth point ξ' must be another point on $f=0$. This pencil of cubics then cuts out a g_2' and the sextic is hyperelliptic and reducible not to a quartic, since the non-singular quartic ($p=3$) does not possess a g_2', but to a quintic with a triple point. (The $F_4(y)=0$ is a double conic.) To show that a hyperelliptic sextic with seven distinct double points does exist, we shall proceed in the following way:

On the hyperboloid $x_1x_2-x_3x_4=0$, where $\dfrac{x_1}{x_3}=\dfrac{x_4}{x_2}=\lambda$ and $\dfrac{x_1}{x_4}=\dfrac{x_3}{x_2}=\mu$, consider the space curve of order 6 which is defined by the general $F(\lambda^2,\mu^4)=0$. If we consider

$$F\equiv\lambda^2\phi_4(\mu,\nu)+\lambda\nu\psi_4(\mu,\nu)+\nu^2\theta_4(\mu,\nu)=0 \tag{1}$$

as the equation of the curve—in fact there is a (1, 1) correspondence between the values of (λ,μ,ν) satisfying 1) and the points of the space sextic—the curve is easily seen to be of genus 3, for (1, 0, 0) is a fourfold point and (0, 1, 0) is a double point. Then any projection of it will be of genus 3. If we connect every point of the curve with a fixed point P not on the hyperboloid, we have a projecting cone of order 6. Any plane section of this cone is a sextic of genus 3. Moreover, the seven double points of this sextic must be distinct, for no line from P can cut the hyperboloid (and consequently the curve) in more than two points. But $F(\lambda^2,\mu^4)$ possessed a g_2' which must be retained by projection. Therefore we have obtained a plane sextic of genus 3

with seven distinct double points, which is hyperelliptic. The canonical form for the hyperelliptic curve of genus 3 is the quintic with a triple point which may be obtained by projecting the space curve from a point on it. The groups of transformations belonging to our hyperelliptic sextic are those belonging to this canonical quintic; these have been discussed by Wiman.*

Case 3) If $f=0$ is to remain pointwise invariant under the Geiser transformation, *i. e.*, if it is the invariant sextic of the transformation, then $\xi=\xi'$ and there is a (1, 1) correspondence between the points of this sextic and the non-singular quartic $F(y)=0$ mentioned above. Then the groups of transformations belonging to the non-hyperelliptic $f_6=0$ are the groups which belong to the non-singular quartic $F(y)=0$, into which the $f_6=0$ is transformed. These groups have also been discussed by Wiman in the paper mentioned above. Corresponding to the linear transformations of points of $F(y)=0$ are the transforms of these collineations, by which pencils of adjoint cubics are sent into pencils of adjoint cubics.

Consequently we have seen that the transformations belonging to any sextic of genus 3 are the conjugates of the transformations belonging to the canonical form of the genus, with respect to the birational transformation which reduces the sextic to the canonical form.

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* A. Wiman, "Ueber die Hyperelliptischen Curven," *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI.